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LETTER TO THE EDITOR

Crossover from percolation to random animals and compact clusters

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Abstract. We introduce a field-like variable to develop a generating function for the percolation problem which, in the appropriate limits, also describes the statistics of random animals (dilute branched polymers) and compact clusters (collapsed branched polymers). In this description, we study the crossover from percolation to random animals and compact clusters using a two-parameter position space renormalisation-group approach. We obtain the global flow diagram in the two-parameter space and calculate the critical properties. We find that the critical behaviour is described below the percolation threshold p_c by the random-animal fixed point, and above p_c by the compact-cluster fixed point. We also propose a Hamiltonian formulation using the Q -state Potts model. We show that the crossover from percolation to random animals can be described by taking a specific limit of the field-like variable.

1. Introduction

Percolation has recently attracted considerable attention because it exhibits all the intricate complexities of second-order phase transitions, and also because it can be used to describe a wide variety of physical systems (see e.g. Stauffer (1979), Essam (1980) and references therein). In addition, percolation theory serves as a simple introduction to cluster-approximation approaches to collective phenomena.

In pure percolation, elements (usually taken to be sites or bonds on a lattice) are placed entirely at random. Each element is occupied with a probability p and is empty with a probability $q = 1 - p$, independent of the occupation of all other elements. A cluster of size s is defined as s elements connected by nearest-neighbour distances. Each finite cluster is terminated by unoccupied elements called perimeter sites. An individual cluster of size s and t perimeter sites is weighted by $p^s q^t$. Accordingly the percolation generating function is defined by

$$F(p, h) = \sum_{s,t} g_{st} p^s q^t \exp(-hs) \quad (1)$$

where g_{st} is the number of geometrically different cluster configurations of s sites with perimeter t . The parameter h is a 'field-like' quantity which couples to every occupied site. The successive derivatives of $F(p, h)$ with respect to h , evaluated at $h = 0$, give the moments of the cluster-size distribution function.

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However, other statistical weightings can be assigned to probe different cluster properties. One such procedure is to weight all configurations of perimeter sites equally—that is, to let $q = 1$ and weight a cluster only by the number of its occupied sites. In this way one obtains the statistics of all possible geometrically distinct s -site clusters. These are usually called *random animals*, because they represent all the possible ‘animals’ that one can ‘create’ out of a fixed number of ‘cells’ (for early references see Domb (1976)). The random-animal generating function can be written as

$$F_a(K) = \sum_{s,t} g_{st} K^s \quad (2)$$

where K is the fugacity, or weight, that is assigned to each site (or bond) on the cluster. The sum $\sum_t g_{st}$ gives the total number of animals of size s , and has been extensively studied by exact enumeration for several common lattices up to nine dimensions (see Gaunt (1980) and references therein). Random animals have also been used as a model for branched polymers in the dilute limit (Lubensky and Isaacson 1979, Family 1980). Another way to weight the clusters is to take the limit $q \rightarrow 0$. Obviously, the ensemble of clusters left are only those with the smallest perimeter, i.e. *compact clusters*. In analogy with linear polymers, compact clusters can be used to describe collapsed branched polymers.

The difference between the three models that we have described is the statistical weights that are given to the clusters. However, we can describe all three models within a unified theory. To do this, we consider (1) and allow h to take on negative, as well as positive, values. Then, we define a new independent fugacity-like parameter K by $K = e^{-h} p$. This definition of K allows us to maintain the probabilistic interpretation for p and q , i.e. $p = 1 - q$, and to study the limit $p \rightarrow 0$ by letting $h \rightarrow -\infty$ with K finite. In this way, (1) can be rewritten as

$$F(K, q) = \sum_{s,t} g_{st} K^s q^t \quad (3)$$

In the limit $q \rightarrow 1$ and K finite, (3) reduces to the animal generating function (2). The limit $q \rightarrow 0$ of (3) describes the statistics of compact clusters.

In this Letter we study the crossover from percolation to random animals and compact clusters in two ways: (1) using a two-parameter position space renormalisation group (PSRG) and (2) using the Q -state Potts model formulation. In the PSRG approach, we obtain the global flow diagram in the two-parameter space and calculate the critical properties. By studying the Q -state Potts model Hamiltonian, we show that the crossover from percolation to random animals can be described by taking a specific limit of the field-like variable.

2. Position space renormalisation group

We use the cell PSRG approach of Reynolds *et al* (1980) for percolation on a triangular lattice. In this method, a cell-to-site renormalisation is carried out using the criterion of finding a connected path spanning the cell. For the triangular lattice the simplest cell consists of three sites. This cell reduces to a single site on the renormalised lattice whose lattice spacing is rescaled by a factor of $b = 3^{1/2}$. Applying the spanning rule to the three-site cell, we see that the cell spans if all three sites are occupied or if any two sites

are present and one is vacant. Thus, the renormalised site probability p' is given by

$$p' = p^3 + 3p^2q. \quad (4)$$

This recursion relation gives an RG transformation for pure percolation in the absence of a field.

To calculate the recursion relation for the fugacity K and the probability q in the generating function (3), we use an analogous renormalisation criterion to define a renormalised fugacity K' by

$$K' = \sum_{[C_{sp}]} g_{st} K^s q^t, \quad (5)$$

where the sum is over all the spanning clusters $[C_{sp}]$ of size s and perimeter t . All spanning clusters are mapped into a single site with a weight K' . Therefore, for the three-site triangular cell we have

$$K' = K^3 + 3K^2q. \quad (6)$$

A similar RG has recently been developed for the study of linear and branched polymers on a square lattice (Family 1980).

To renormalise the probability q , we use the relation $q = 1 - p$ in (4) and find

$$q' = q^3 + 3q^2(1 - q). \quad (7)$$

The coupled recursion relations (6) and (7) constitute a two-parameter RG transformation for the percolation problem in the presence of a field. These equations may be solved numerically for the fixed points, critical surface and critical exponents. The global flow diagram is shown in figure 1. From (6) and (7) we find the fixed points of the transformation at $(K' = K = K^*, q' = q = q^*)$. These equations have nine fixed points as shown in figure 1. The three critical fixed points of interest are indicated by solid dots on the diagram and their values are listed in table 1.

The most unstable fixed point is the percolation fixed point. Starting near this fixed point, the flow on the critical surface is either to the animal fixed point (at $q = 1$) or to the

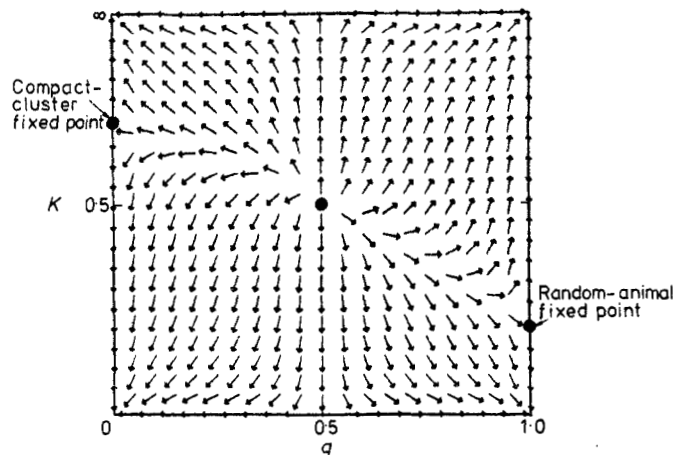


Figure 1. The renormalisation-group flow diagram from equations (6) and (7). The three fixed points of interest are shown as solid dots. The fixed point at the centre of the diagram is the percolation fixed point. Random-animal and compact-cluster fixed points are explicitly indicated on the diagram.

Table 1. Summary of the PSRG results for the three fixed points of interest. Best estimates for the exponents are given in parentheses.

Fixed point	K^*	q^*	ν_K	ν_p
Random animal	0.3028	1	0.7444 (0.65 ± 0.02 ^a)	—
Percolation	$\frac{1}{2}$	$\frac{1}{2}$	0.6774 (0.53 ± 0.02 ^a)	1.3547 (1.3333 ^b)
Compact cluster	1	0	$\frac{1}{2}$ (Exact)	—

^a Stauffer and Holl (1980, to be published).^b Eschbach *et al* (1980).

compact-cluster fixed point (at $q = 0$). Note that all the critical behaviour below p_c is governed by the animal fixed point, and all critical behaviour above p_c is controlled by the compact-cluster fixed point.

The eigenvalues of the linearised recursion relations are obtained as usual by computing the eigenvalues of the matrix

$$T_{\alpha\beta} = \partial\alpha'/\partial\beta \quad \text{where } \alpha, \beta = K \text{ or } q, \quad (8)$$

evaluated at the critical fixed points. From (7) we see that $\partial q'/\partial K = 0$, so that the eigenvalues are

$$\lambda_K = 3K^2 + 6Kq|_{K^*,q^*} \quad \text{and} \quad \lambda_q = -6q^2 + 6q|_{K^*,q^*}. \quad (9)$$

The correlation length exponent ν_p for percolation is given by $\nu_p = \ln(b)/\ln(\lambda_q)$, and the exponent ν_K is given by $\nu_K = \ln(b)/\ln(\lambda_K)$. Using scaling arguments (Stauffer 1979, Essam 1980) this exponent can be related to the cluster size s by

$$\xi \sim s^{\nu_K} \quad (s \rightarrow \infty) \quad (10)$$

where ξ is the mean end-to-end length of the cluster. The results for the exponents at the three critical fixed points of interest are listed in table 1.

The exponent ν_K has been determined by Stauffer (1979) (who calls it $\rho(p)$) and by Stauffer and Holl (1980, to be published), using Monte Carlo and series techniques. They find that ν_K varies continuously with p . This is an artifact of the numerical methods. Using the RG equations, we find only three critical behaviours, and therefore ν_K has only three values. This implies that large finite clusters have three different degrees of ramification for $p < p_c$, $p = p_c$, and $p > p_c$ †. In the language of fractals (Mandelbrot (1977); see also § 4.3 in Stauffer (1979)), this indicates that clusters below p_c have the same fractal dimension as random animals, and above p_c clusters are fully compact and the fractal dimension of the clusters equals the Euclidean dimension. We can show that in any dimension d the RG transformation gives $\nu_K = 1/d$ for compact

† Note that, although below and above p_c the probability of having a large finite cluster is negligible, one can nevertheless study the statistics of a cluster of any size s by considering a restricted ensemble where only one single cluster of s elements is present. A configuration of such a cluster is then weighted by $p^s q^i / p^s \sum_{\text{config}} q^i = q^i / \sum_{\text{config}} q^i$.

clusters. Consider a cell of size b^d in d dimensions. In the limit $q \rightarrow 0$, the recursion relation is given by $K' = K^{b^d}$ so that $K^* = 1$, $\lambda_K = b^d$ and $\nu_K \equiv 1/d$.

The exponent ν_K at the percolation threshold is related to the usual percolation exponents by (Stauffer 1979)

$$\nu_K = \nu_p / (\beta_p \delta_p) \quad (p = p_c). \tag{11}$$

Therefore, the crossover exponent ϕ is given by

$$\phi = \nu_K / \nu_p = \Delta^{-1}, \tag{12}$$

where Δ is the gap exponent. Using the numerical values of ν_K and ν_p from table 1, we find $\phi = 0.5$. This result can be compared with the best estimate for Δ^{-1} which is 0.39 (Stauffer 1979).

Note that our PSRG approach is general and is not limited to the site problem, the triangular lattice, small cells, or the dimensionality of the system. One can follow the same procedure for larger cells, on any lattice, by generalising the recursion relations (7) and (8). In that case, the overall picture of the flow diagram should remain unchanged. However, by going to larger cells (Reynolds *et al* 1980) one should be able to obtain very accurate estimates for the exponents and other critical properties.

3. Hamiltonian formulation

We now give a Hamiltonian formulation for the bond model. Consider the following Q -state Potts Hamiltonian (Potts 1952) in an external field, on a regular d -dimensional lattice:

$$-\beta \mathcal{H} = J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} + h \sum_{\langle ij \rangle} \delta_{\sigma_i 1 \sigma_j 1}. \tag{13}$$

σ_i are Potts variables which can assume Q values $\sigma_i = 1, 2, \dots, Q$. It has been shown by Wu (1978) that the partition function for this Hamiltonian can be written as

$$Z = \sum_{\{C\}} p^{|C|} q^{|D|} \prod_{\text{clusters}} (e^{hn_B} + s - 1), \tag{14}$$

where $q = 1 - p = e^{-J}$, C is a configuration of bonds in the set of all the bonds E , $D = E - C$ and $|C|$ and $|D|$ are the number of bonds in the subsets C and D . The product is over all the clusters in the configuration C and n_B is the number of bonds in a cluster. From (14) we can construct the generating function (1) for the bond problem by taking the limit $s \rightarrow 1$ of $d \lg Z/ds$.

If we restrict ourselves to small and positive h we obtain the percolation problem. It is well known (Kasteleyn and Fortuin 1974) that percolation is described by the Potts model in the limit $s \rightarrow 1$. To obtain the random-animal problem, we allow h to take on negative values and then take the limit $h \rightarrow -\infty$, $q \rightarrow 1$, $pe^{-h} = K$, with K finite. Therefore, the crossover from percolation to lattice animals can be studied in a Hamiltonian formalism by using (13) in the special limits that we have described.

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Note added in proof. After this work was submitted for publication we received two preprints from A B Harris and T C Lubensky in which the connection between percolation and lattice animals was discussed. Using a field theory approach they find that the percolation point is a multi-critical point; in agreement with our result.

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